1 Problem Solving: Secant/Tangent Lines and Average/Instantaneous Velocity

In these types of problems, there are a set of steps we use to complete them:

1. **Read the problem and identify the desired result(s).** Underline or circle the quantity or quantities desired.
2. **Review how to achieve the desired outcome.** This means that before you can actually compute or get to your desired outcome, what OTHER quantities are needed? For example, before you can compute the instantaneous velocity you need to know the average velocity in terms of an unknown quantity.
3. **Look for given values of your independent variable (usually x or t in these problems) and compute their other component via the given expression.** If you are given two values of x and an expression involving x to find y, compute y; the same goes for t and $s(t)$.
4. **If only given one value of your independent variable, choose your other value of the independent variable to involve an unknown quantity.** Many times in these problems, you will be given one value of x or t and your job is to find the other. To do so, you have two options
   - Select the second value to be the first value plus some "very small" value. For example if your first value is $x = 1$, you can choose your second value to be $x = 1 + h$ where $h > 0$, but h is very small.
   - Select the second value to be an arbitrary unknown value u that is greater than the first value, but will eventually get closer and closer to the first value. This means that if your first value was $x = 2$ for example, you can pick your second value to be $x = u$ where $u > 2$
5. **If there is an unknown quantity from step 4, determine the value that you want it to approach.** This will be your limit value later on in the problem. This value is typically your first value of x, t, or other independent variable as you want the secant line to get closer and closer to this original point as to create a tangent line!
6. **With all this computed and ready for our use, compute the slope of the secant line (or average velocity):**

$$m_S = \frac{y_2 - y_1}{x_2 - x_1}$$

   or if you are calculating average velocity, v_{avg}, with a given position function $s(t)$ and various times t_2 and t_1,

$$v_{avg} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

7. **Using the limit value from step 5, find the tangent slope or instantaneous velocity by**

$$m_T = \lim_{quantity \to limitvalue} m_S$$

$$v_{inst} = \lim_{quantity \to limitvalue} v_{avg}$$

8. **If a "line" is required, use the first point given, the appropriate slope and the point-slope form of a line,** $y - y_1 = m(x - x_1)$ **to find the line. Simplify if desired.**

Note: Depending on the construction of the problem, or what is being asked in the problem, you may need to skip various steps as they do not apply.

2 Examples

Example 1: Compute the tangent line of the curve $y = x^2 - 8x + 4$ at $x = 3$.

Solution: Let's go through the problem solving process discussed on the previous page:

1. After reading this, we want to find the tangent line.

2. In order to compute the tangent line, we need the tangent slope, but before that we need to find the average velocity as the tangent slope requires the average velocity

3. We are given $x = 3$, and the corresponding value of y is $y = (3)^2 - 8(3) + 4 = 9 - 24 + 4 = -11$, so we have the point $(3, -11)$.

4. As we are only given one value of x, we need to select another value. As mentioned, we can select this in one of two ways: (both are mathematically correct)

   - Let the second value be $x = 3 + h$ where $h > 0$. If we do this, our limit value will be 0 as we want h to get closer and and closer to $x = 3$ as the problem is asking for the tangent line at this value.
   - Let the second value be $x = u$ where $u > 3$. Doing this would mean that the limit value would be 3 for the same reasons as previously mentioned, but this time we are not adding anything onto the initial value; just taking values closer and closer to $x = 3$.

5. In doing step 4, these values were decided.

6. To prove both methods are equal, we will show each. By the first method, we get the second y value to be $y = (3+h)^2 - 8(3+h) + 4 = h^2 + 6h + 9 - 24 - 8h + 4 = h^2 - 2h - 11$, hence the secant slope is:

$$m_S = \frac{h^2 + 14h - 11 - (-11)}{3 + h - 3} = \frac{h^2 - 2h}{h} = h - 2$$

   Meanwhile by the second method, our second y-value would be $y = u^2 - 8u + 4$ and so our secant slope is:

$$m_S = \frac{u^2 - 8u + 4 - (-11)}{u - 3} = \frac{u^2 - 8u + 15}{u - 3} = \frac{(u-5)(u-3)}{u-3} = u - 5$$

7. As we have the appropriate limit values and the secant slope, we can find the tangent slope both ways:

$$m_T = \lim_{h \to 0} (h - 2) = 0 - 2 = -2$$

   and the by the other method,

$$m_T = \lim_{u \to 3} (u - 5) = 3 - 5 = -2$$

8. Now using the tangent slope we just calculated and the point $(3, -11)$, when we put it into the point-slope equation, we get:

$$y - 3 = (-2)(x + 11)$$

   and simplifying, we get

$$y = -2x - 19$$

Example 2: A paper glider's position (in meters) is modeled by the equation $s(t) = -5t^2 + 7t + 11$ at time t seconds. Find the instantaneous velocity the paper glider is experiencing at $t = 2$ seconds.

Solution: Again we go through the problem solving process:

1. We are to compute the instantaneous velocity of the paper glider.
2. To do step one, we need to know the average velocity, and with this we need to know a second time value.
3. When $t = 2$, $s(2) = -5(2)^2 + 7(2) + 11 = -20 + 14 + 11 = 5$.
4. The first method mentioned in this step will be omitted and is left to the reader. So we select our other time to be $t = u$ and so $s(u) = -5u^2 + 7u + 11$
5. Our intention in selecting u was that $u > 2$, but also we want u to approach 2 as we are asked for the instantaneous velocity at this point.
6. With the information we have found, we can compute the average velocity:

   $$v_{avg} = \frac{-5u^2 + 7u + 11 - (5)}{u - 2} = \frac{-5u^2 + 7u + 6}{u - 2}$$

   We omit the synthetic or long division that takes place with $u - 2$, but the reader can verify the following:

   $$v_{avg} = \frac{(-5u - 3)(u - 2)}{u - 2} = -5u - 3$$

7. So the instantaneous velocity of the object is:

   $$v_{inst} = \lim_{u \to 2}(-5u - 3) = -5(2) - 3 = -13$$

   So when $t = 2$ seconds, the glider is experiencing an instantaneous velocity of -13 m/s.

8. Not applicable